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# ASYMPTOTIC PROPERTY OF PERTURBED NONLINEAR SYSTEMS

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Abstract. In this paper, we show that the solutions to perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t))$$

have asymptotic property by imposing conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), T_1 y(s))ds, h(t, y(t), T_2 y(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t, y)$.

## 1. Introduction

Elaydi and Farran[8] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of uniformly Lipschitz stable. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Pachpatte[18,19] studied the stability and asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term. Gonzalez and Pinto[9] investigated the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Choi et al.[5,6] examined Lipschitz and exponential asymptotic stability for nonlinear functional systems. Also, Goo [10,12,13] and Goo et al.[14,15] investigated Lipschitz and asymptotic stability for perturbed differential systems.

In this paper, we investigate asymptotic behavior for solutions of perturbed nonlinear systems using integral inequalities. The method incorporating integral inequalities takes an important place among the

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methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations.

## 2. Preliminaries

We consider the unperturbed nonlinear system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean $n$-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed functional differential system of (2.1)

(2.2)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s)) ds + h(t, y(t), T_2 y(t)),\ y(t_0) = y_0,$$

where $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ , $g(t, 0, 0) = h(t, 0, 0) = 0$, and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are continuous operators . The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. For an $n \times n$ matrix $A$, define the norm $|A|$ of $A$ by $|A| = \sup_{|x| \le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0) v(t),\ v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0)) z(t),\ z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

We give some of the main definitions that we need in the sequel[8].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \le M|x_0|$ whenever $|x_0| \le \delta$ and $t \ge t_0 \ge 0$, (EAS) *exponentially asymptotically stable* if there exist constants $K > 0$ , $c > 0$, and $\delta > 0$ such that

$$|x(t)| \le K\,|x_0|e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \delta$,

(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \le K\,e^{-c(t-t_0)}, 0 \le t_0 \le t$$

provided that $|x_0| < \infty$.

Remark 2.2. [9] The last definition implies that for $|x_0| \le \delta$

$$|x(t)| \le K\,|x_0|e^{-c(t-t_0)}, 0 \le t_0 \le t.$$

We begin by recalling some preliminary results.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point $(t_0, y_0)$ in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.3. [2] *Let $x$ and $y$ be a solution of (2.1) and (2.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \ge t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Lemma 2.4. *(Bihari-type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in $u$. Suppose that, for some $c > 0$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda(s)w(u(s))ds,\ t \ge t_0 \ge 0.$$

*Then*

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s)ds\Big],$$

*where* $t_0 \le t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.5. [3] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* $u$, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds \\ &+ \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds \\ &+ \int_{t_0}^{t} \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds, \quad 0 \le t_0 \le t. \end{aligned}$$

*Then*

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau \\ +\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau\Big)ds\Big], \end{aligned}$$

*where* $t_0 \le t < b_1$, $W$, $W^{-1}$ *are the same functions as in Lemma 2.4, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau \\ +\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau\Big)ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

Lemma 2.6. [16] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10} \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* $u$, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^{t}\lambda_1(s)u(s)ds + \int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau)) \\ &+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)u(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)w(u(r))dr\Big)d\tau ds \\ &+ \int_{t_0}^{t}\lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)w(u(\tau))d\tau ds. \end{aligned}$$

*Then, we have*

$$\begin{aligned} u(t) &\le W^{-1}\Big[W(c)+\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau)+\lambda_4(\tau)\\ &+ \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr+\lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr\Big)d\tau\\ &+ \lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)d\tau)ds\Big], \end{aligned}$$

*where* $t_0 \le t < b_1$, $W$, $W^{-1}$ *are the same functions as in Lemma 2.4, and*

$$\begin{aligned} b_1 = \sup\Big\{&t \ge t_0 : W(c)+\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)\\ &+\lambda_4(\tau)+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr\\ &+\lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr)d\tau+\lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

We need the following corollary for the proof.

Corollary 2.7. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* $u$, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^{t}\lambda_1(s)u(s)ds+\int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau)u(\tau)+\lambda_4(\tau)w(u(\tau))\\ &+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)u(r)dr+\lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)w(u(r))dr\Big)d\tau ds. \end{aligned}$$

*Then, we have*

$$\begin{aligned} u(t) \le W^{-1}\Big[&W(c)+\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}\Big(\lambda_3(\tau)+\lambda_4(\tau)\\ &+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr+\lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr\Big)d\tau)ds\Big], \end{aligned}$$

*where* $t_0 \le t < b_1$, $W$, $W^{-1}$ *are the same functions as in Lemma 2.4, and*

$$\begin{aligned} b_1 = \sup\Big\{&t \ge t_0 : W(c)+\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)+\lambda_4(\tau)\\ &+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr+\lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr)d\tau)ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

LEMMA 2.8. [11] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10} \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* $u$, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds \\ &+ \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \Big( \lambda_3(\tau) u(\tau) + \lambda_4(\tau) w(u(\tau)) \\ &+ \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r) u(r) dr + \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r) w(u(r)) dr \Big) d\tau ds \\ &+ \int_{t_0}^{t} \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau) w(u(\tau)) d\tau ds. \end{aligned}$$

*Then, we have*

$$\begin{aligned} u(t) \leq W^{-1}\Big[ W(c) &+ \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \Big( \lambda_3(\tau) + \lambda_4(\tau) \\ &+ \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r) dr + \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r) dr \Big) d\tau \\ &+ \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau) d\tau) ds \Big], \end{aligned}$$

*where* $t_0 \leq t < b_1$, $W$, $W^{-1}$ *are the same functions as in Lemma 2.4, and*

$$\begin{aligned} b_1 = \sup \Big\{ t \geq t_0 : W(c) &+ \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) \\ &+ \lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r) dr \\ &+ \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r) dr) d\tau + \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau) d\tau) ds \in \mathrm{dom W^{-1}} \Big\}. \end{aligned}$$

For the proof, we need the following corollary .

COROLLARY 2.9. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* $u$, $u \leq w(u)$. *Suppose that* *for some $c > 0$ and $0 \le t_0 \le t$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \Big(\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau)) + \lambda_5(\tau)\int_{t_0}^{\tau} \lambda_6(r)u(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau} \lambda_8(r)w(u(r))dr\Big)d\tau ds.$$

*Then, we have*

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \Big(\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau} \lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau} \lambda_8(r)dr\Big)d\tau)ds\Big],$$

*where $t_0 \le t < b_1$, $W$, $W^{-1}$ are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau} \lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau} \lambda_8(r)dr)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

## 3. Main results

In this section, we investigate asymptotic behavior for solutions of the perturbed functional differential systems.

To obtain asymptotic behavior, the following assumptions are needed:

(H1) The solution $x = 0$ of (2.1) is EASV.

(H2) $w(u)$ is nondecreasing in $u$ such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$.

THEOREM 3.1. *Suppose that (H1), (H2), and that the perturbing term $g(t, y, T_1 y)$ satisfies*

$$|g(t, y(t), T_1 y(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|\Big), \tag{3.1}$$

$$|T_1 y(t)| \le c(t)\int_{t_0}^{t} k(s)|y(s)|ds + d(t)\int_{t_0}^{t} m(s)w(|y(s)|)ds, \tag{3.2}$$

$$|h(t,y(t),T_2y(t))| \le \int_{t_0}^{t} e^{-\alpha s}p(s)|y(s)|ds + |T_2y(t)|, \tag{3.3}$$

*and*

$$|T_2y(t)| \le e^{-\alpha t}n(t)|y(t)| + \int_{t_0}^{t} e^{-\alpha s}q(s)w(|y(s)|)ds, \tag{3.4}$$

*where* $\alpha > 0,\ a,b,c,d,k,m,n,p,q,w \in C(\mathbb{R}^+),\ a,b,c,d,k,m,n,p,q \in L^1(\mathbb{R}^+)$. *If*

(3.5)

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}\Big(n(s) + e^{\alpha s}\int_{t_0}^{s}(a(\tau)+b(\tau) \\ &+ p(\tau)+q(\tau)+c(\tau)\int_{t_0}^{\tau}k(r)dr + d(\tau)\int_{t_0}^{\tau}m(r)dr)d\tau\Big)ds\Big] < \infty, \end{aligned}$$

*where* $t \ge t_0$, $c = |y_0|Me^{\alpha t_0}$, *and* $W$, $W^{-1}$ *are the same functions as in Lemma 2.4, then all solutions of (2.2) approach zero as* $t \to \infty$.

*Proof.* Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), it is EAS by remark 2.2. Using Lemma 2.3, together with (3.1), (3.2), (3.3), and (3.4), we obtain

$$\begin{aligned} |y(t)| \le& M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(\int_{t_0}^{s} e^{-\alpha\tau}((a(\tau)+p(\tau))|y(\tau)| \\ &+(b(\tau)+q(\tau))w(|y(\tau)|) + c(\tau)\int_{t_0}^{\tau}k(r)|y(r)|dr \\ &+d(\tau)\int_{t_0}^{\tau}m(r)w(|y(r)|)dr)d\tau + e^{-\alpha s}n(s)|y(s)|\Big)ds. \end{aligned}$$

Applying the assumption (H2), we have

$$\begin{aligned} |y(t)| \le& M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big(e^{\alpha s}\int_{t_0}^{s}((a(\tau)+p(\tau))|y(\tau)|e^{\alpha\tau} \\ &+(b(\tau)+q(\tau))w(|y(\tau)|e^{\alpha\tau}) + c(\tau)\int_{t_0}^{\tau}k(r)|y(r)|e^{\alpha r}dr \\ &+d(\tau)\int_{t_0}^{\tau}m(r)w(|y(r)|e^{\alpha r})dr)d\tau + n(s)|y(s)|e^{\alpha s}\Big)ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. An application of Corollary 2.7 and (3.5) obtains

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(n(s) + e^{\alpha s}\int_{t_0}^{s}(a(\tau) + b(\tau) + p(\tau) + q(\tau) + c(\tau)\int_{t_0}^{\tau}k(r)dr + d(\tau)\int_{t_0}^{\tau}m(r)dr)d\tau)\Big)ds\Big] \le e^{-\alpha t}M(t_0),$$

where $t \ge t_0$ and $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (2.2) approach zero as $t \to \infty$. □

Remark 3.2. Letting $n(t) = p(t) = q(t) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.5 in [4].

Theorem 3.3. *Suppose that (H1), (H2), and that the perturbing term $g(t, y, T_1y)$ satisfies*

$$\int_{t_0}^{t}|g(s, y(s), T_1y(s))|ds \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|\Big), \tag{3.6}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^{t}k(s)w(|y(s)|)ds + d(t)\int_{t_0}^{t}m(s)|y(s)|ds, \tag{3.7}$$

$$|h(t, y(t), T_2y(t))| \le e^{-\alpha t}(c(t)w(|y|) + |T_2y(t)|), \tag{3.8}$$

*and*

$$|T_2y(t)| \le q(t)|y(t)| + d(t)\int_{t_0}^{t}p(s)|y(s)|ds, \tag{3.9}$$

*where* $\alpha > 0$, $a, b, c, d, k, m, p, q, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, p, q \in L_1(\mathbb{R}^+)$. *If*

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}\Big(a(s) + b(s) + c(s) + q(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau + d(s)\int_{t_0}^{s}(m(\tau) + p(\tau))d\tau\Big)ds\Big] < \infty, \tag{3.10}$$

*where $b_1 = \infty$, $c = M|y_0|e^{\alpha t_0}$, and $W$, $W^{-1}$ are the same functions as in Lemma 2.4, then all solutions of (2.2) approach zero as $t \to \infty$.*

*Proof.* Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), it is EAS. Using

Lemma 2.3, together with (3.6), (3.7), (3.8), and (3.9), we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(e^{-\alpha s}(a(s)|y(s)| + b(s)w(|y(s)|) \\ +b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau + d(s)\int_{t_0}^{s}(m(\tau)+p(\tau))|y(\tau)|d\tau \\ +q(s)|y(s)| + c(s)w(|y(s)|)\Big)ds.$$

It follows from (H2) that

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big((a(s)+q(s))|y(s)|e^{\alpha s} \\ +(b(s)+c(s))w(|y(s)|e^{\alpha s}) + b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau \\ +d(s)\int_{t_0}^{s}(m(\tau)+p(\tau))|y(\tau)|e^{\alpha\tau}d\tau\Big)ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, an application of Lemma 2.5 and (3.10) obtains

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s)+b(s)+c(s)+q(s) \\ +b(s)\int_{t_0}^{s} k(\tau)d\tau + d(s)\int_{t_0}^{s}(m(\tau)+p(\tau))d\tau\Big)ds\Big] \le e^{-\alpha t}M(t_0),$$

where $c = M|y_0|e^{\alpha t_0}$. Therefore, all solutions of (2.2) approach zero as $t \to \infty$. □

REMARK 3.4. Letting $c(t) = d(t) = q(t) = 0$ in Theorem 3.3, we obtain the same result as that of Theorem 3.4 in [15].

THEOREM 3.5. *Suppose that (H1), (H2), and that the perturbing term $g(t, y, T_1y)$ satisfies*

$$|g(t, y(t), T_1y(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|\Big), \tag{3.11}$$

$$|T_1y(t)| \le c(t)\int_{t_0}^{t} k(s)|y(s)|ds + d(t)\int_{t_0}^{t} m(s)w(|y(s)|)ds, \tag{3.12}$$

$$|h(t, y(t), T_2y(t))| \le \int_{t_0}^{t} e^{-\alpha s}p(s)|y(s)|ds + |T_2y(t)|, \tag{3.13}$$

*and*

$$|T_2y(t)| \le e^{-\alpha t}n(t)w(|y(t)|) + \int_{t_0}^{t} e^{-\alpha s}q(s)w(|y(s)|)ds, \tag{3.14}$$

*where* $\alpha > 0$, $a, b, c, d, k, m, n, p, q, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, n, p, q \in L^1(\mathbb{R}^+)$. *If*

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}\Big(n(s) + e^{\alpha s}\int_{t_0}^{s}(a(\tau) + b(\tau) \\ &+ p(\tau) + q(\tau) + c(\tau)\int_{t_0}^{\tau}k(r)dr + d(\tau)\int_{t_0}^{\tau}m(r)dr)d\tau\Big)ds\Big] < \infty, \end{aligned} \tag{3.15}$$

*where* $t \ge t_0$, $c = |y_0|Me^{\alpha t_0}$, *and* $W$, $W^{-1}$ *are the same functions as in Lemma 2.4, then all solutions of (2.2) approach zero as* $t \to \infty$.

*Proof.* Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), it is EAS by remark 2.2. Using Lemma 2.3, together with (3.11), (3.12), (3.13), and (3.14), we obtain

$$\begin{aligned} |y(t)| \le& M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(\int_{t_0}^{s} e^{-\alpha\tau}((a(\tau) + p(\tau))|y(\tau)| \\ &+(b(\tau) + q(\tau))w(|y(\tau)|) + c(\tau)\int_{t_0}^{\tau}k(r)|y(r)|dr \\ &+d(\tau)\int_{t_0}^{\tau}m(r)w(|y(r)|)dr)d\tau + e^{-\alpha s}n(s)w(|y(s)|)\Big)ds. \end{aligned}$$

Applying the assumption (H2), we have

$$\begin{aligned} |y(t)| \le& M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big(e^{\alpha s}\int_{t_0}^{s}((a(\tau) + p(\tau))|y(\tau)|e^{\alpha\tau} \\ &+(b(\tau) + q(\tau))w(|y(\tau)|e^{\alpha\tau}) + c(\tau)\int_{t_0}^{\tau}k(r)|y(r)|e^{\alpha r}dr \\ &+d(\tau)\int_{t_0}^{\tau}m(r)w(|y(r)|e^{\alpha r})dr)d\tau + n(s)w(|y(s)|e^{\alpha s})\Big)ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. An application of Corollary 2.9 and (3.15) obtains

$$\begin{aligned} |y(t)| \le& e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(n(s) + e^{\alpha s}\int_{t_0}^{s}(a(\tau) + b(\tau) + p(\tau) \\ &+q(\tau) + c(\tau)\int_{t_0}^{\tau}k(r)dr + d(\tau)\int_{t_0}^{\tau}m(r)dr)d\tau\Big)ds\Big] \le e^{-\alpha t}M(t_0), \end{aligned}$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (2.2) approach zero as $t \to \infty$. $\square$

REMARK 3.6. Letting $n(t) = p(t) = q(t) = 0$ in Theorem 3.5, we obtain the similar result as that of Theorem 3.5 in [4].

THEOREM 3.7. *Suppose that (H1), (H2), and that the perturbing term $g(t, y, T_1 y)$ satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds \leq e^{-\alpha t}\Big(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|\Big), \tag{3.16}$$

$$|T_1 y(t)| \leq b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds + d(t)\int_{t_0}^{t} p(s)|y(s)|ds, \tag{3.17}$$

$$|h(t, y(t), T_2 y(t))| \leq e^{-\alpha t}(c(t)w(|y|) + |T_2 y(t)|), \tag{3.18}$$

*and*

$$|T_2 y(t)| \leq q(t)|y(t)| + b(t)\int_{t_0}^{t} m(s)w(|y(s)|)ds, \tag{3.19}$$

*where $\alpha > 0$, $a, b, c, d, k, m, p, q, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, p, q \in L_1(\mathbb{R}^+)$. If*

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}\Big(a(s) + b(s) + c(s) + q(s) \\ &+ b(s)\int_{t_0}^{s}(k(\tau) + m(\tau))d\tau + d(s)\int_{t_0}^{s} p(\tau)d\tau\Big)ds\Big] < \infty, \end{aligned} \tag{3.20}$$

*where $b_1 = \infty$, $c = M|y_0|e^{\alpha t_0}$, and $W$, $W^{-1}$ are the same functions as in Lemma 2.4, then all solutions of (2.2) approach zero as $t \to \infty$.*

*Proof.* Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), it is EAS. Using Lemma 2.3, together with (3.16), (3.17), (3.18), and (3.19), we have

$$\begin{aligned} |y(t)| \leq& M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(e^{-\alpha s}(a(s)|y(s)| + b(s)w(|y(s)|) \\ &+ b(s)\int_{t_0}^{s}(k(\tau) + m(\tau))w(|y(\tau)|)d\tau + d(s)\int_{t_0}^{s} p(\tau))|y(\tau)|d\tau \\ &+ q(s)|y(s)| + c(s)w(|y(s)|)\Big)ds. \end{aligned}$$

It follows from (H2) that

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big((a(s)+q(s))|y(s)|e^{\alpha s}$$

$$+(b(s)+c(s))w(|y(s)|e^{\alpha s}) + b(s)\int_{t_0}^{s}(k(\tau)+m(\tau))w(|y(\tau)|e^{\alpha\tau})d\tau$$

$$+d(s)\int_{t_0}^{s} p(\tau)|y(\tau)|e^{\alpha\tau}d\tau\Big)ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, an application of Lemma 2.5 and (3.20) obtains

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s)+b(s)+c(s)+q(s)$$

$$+b(s)\int_{t_0}^{s}(k(\tau)+m(\tau))d\tau + d(s)\int_{t_0}^{s}p(\tau)d\tau\Big)ds\Big] \le e^{-\alpha t}M(t_0),$$

where $c = M|y_0|e^{\alpha t_0}$. Therefore, all solutions of (2.2) approach zero as $t \to \infty$. □

Remark 3.8. Letting $a(t) = c(t) = k(t) = m(t) = q(t) = 0$ in Theorem 3.7, we obtain the same result as that of Theorem 3.3 in [15].

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